March 1992

APPENDIX

Alifanov Iterative Regularization Algorithm

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The iterative regularization algorithm has been championed by Alifanov and Rumyantsev in the USSR. It was orginally proposed by Alifanov and is widely used in the Soviet Union. It can be used both for estimating the surface heat flux and thermal properties as functions. There are several variations of the method, depending, for example, if the steepest descent or conjugate gradient methods are used. The method is not easy to extract from the books and papers. The purpose of this appendix is to document the algorithm and give the details in the order in which they can be readily programmed. P. Lamm contributed to our understanding of this problem and derived the basic equations which are given below.

The problem to be solved is

$$k \frac{\partial^2 T}{\partial x^2} = \rho c \frac{\partial T}{\partial t}, \quad 0 < x < L,\ 0 < t < t_f \qquad \text{(A-1)}$$

$$-k \frac{\partial T(0,t)}{\partial x} = q_o(t) = ? \qquad \text{(A-2)}$$

$$-k \frac{\partial T(L,t)}{\partial x} = q_L(t), \text{ known} \qquad \text{(A-3)}$$

$$T(x,0) = T_o(x) \tag{A-4}$$

$$T(d,t) = Y(t) \tag{A-5}$$

The temperature is measured at location $x = d$ and is denoted $Y(t)$. For simplicity, however, the known heat flux at $x = L$ is assumed to be zero and $T_o(x)$ is set equal to a constant, which is taken to be zero.

Three problems are solved at each iteration. First, eqs. (A-1) to (A-4) are solved with $q_o(t)$ replaced with its estimated function $q^{(n)}(t)$ for the nth iteration. Then the adjoint and sensitivity problems must be solved. The adjoint problem is

$$k\frac{\partial^2\psi}{\partial x^2} = -\rho c\frac{\partial\psi}{\partial t} - \delta(x - d)[T(x,t;q(t)) - Y(t)], \; 0 < x < L, \; t = t_f \text{ to } 0 \tag{A-6}$$

$$\frac{\partial\psi(0,t)}{\partial x} = \frac{\partial\psi(L,t)}{\partial x} = 0 \tag{A-7,8}$$

$$\psi(x,t_f) = 0 \tag{A-9}$$

Notice that this adjoint problem goes backward in time and starts at time t_f, the final time. The driving term in the $\psi(x,t)$ problem is the difference between the temperature calculated in the eq. (A-1) to (A-5) problem and the measured temperature, $Y(t)$. The units of ψ are $m^2\text{-}K^2/W$.

The sensitivity problem is the solution of

$$k\frac{\partial^2\theta}{\partial x^2} = \rho c\frac{\partial\theta}{\partial t},\ 0 < x < L,\ 0 < t < t_f \tag{A-10}$$

$$- k\ \frac{\partial\theta\ (0,t)}{\partial x} = p^{(n)}(t) \tag{A-11a}$$

$$\frac{\partial\theta(L,t)}{\partial x} = 0 \tag{A-11b}$$

$$\theta(x,0) = 0 \tag{A-12}$$

where $p^{(n)}(t)$ comes from the solution of the adjoint problem and is described in step 7 below. The units of p(t) are the same as those of ψ, namely m^2-K^2/W. The units of θ are K^3-m^4/W^2.

The above quantities are used in the approach to a minimum of the function

$$S(q(t)) = \int_0^{t_f} [Y(t) - T(d,t)]^2 dt \tag{A-13a}$$

Implicit in the iterative procedure is that S(•) is not precisely minimized but is reduced to the level where it is just less than δ^2, or

$$S^{(n)} \leq \delta^2 \tag{A-13b}$$

where the superscript n refers to the iteration and δ^2 is a measure of the errors in the temperature measurements, Y(t). Other stopping criteria are suggested in the 1985 paper by Alifanov and Balashova. The regularization in the procedure for finding the function q(t) is incorporated in the natural "viscosity" or slowness in the approach to the minimum provided by the methods of steepest descent or conjugate gradient. The latter method converges much more rapidly than the steepest descent method.

The procedure is now outlined in the form of an algorithm.

0. The geometry (L), properties (k, ρ and c), initial condition ($T_o(x)$) measured temperature (Y(t)) and δ^2 are given.

Set n = 0 and start with an estimate of q(t). Usually $q^{(0)}(t) = 0$ is chosen.

1. Set n = n + 1. Solve the temperature problem using eqs. (A-1) to (A-4) with $q_o(t)$ replaced by the estimated function $q^{(n)}(t)$.

Calculate

$$S^{(n)}(t_f) = \int_0^{t_f} [Y(t) - T^{(n)}(d,t)]^2 dt \qquad \text{(A-14a)}$$

If

$$S^{(n)} < \delta^2 \qquad \text{(A-14b)}$$

terminate the computations.

2. Solve the adjoint problem for $\psi(x,t)$, remembering to start at $t = t_f$ and then working backward to $t = 0$. Use eqs. (A-6) to (A-9). The derivative of S with respect to the $q^{(n)}(t)$ function is denoted $S'^{(n)}(t)$ and happens to be equal to the adjoint variable evaluated at $x = 0$,

$$\nabla S^{(n)}(t) = \psi(0,t;q^{(n)}(t)) \qquad \text{(A-15)}$$

3a. If $n = 1$, set

$$\gamma^{(1)} = 0 \qquad \text{(A-16)}$$

and go to step 5.

3b. If $n \geq 2$, calculate

$$\gamma_N^{(n)} = \int_0^{t_f} \psi^{(n)}(t) \ [\psi^{(n-1)}(t) - \psi^{(n)}(t))]dt \qquad \text{(A-17)}$$

where all the auguments of ψ are omitted for convenience and where the N subscript denotes numerator.

4. If $n \geq 2$, calculate

$$\gamma^{(n)} = - \gamma_N^{(n)} \ / \ \gamma_D^{(n-1)} \qquad \text{(A-18)}$$

where $\gamma_D^{(n-1)}$ is obtained from step 10 of the previous iteration. (The D denotes denominator.)

5a. For n = 1 or for the steepest descent method, use

$$p^{(n)}(t) = \psi(0,t;q^{(n)}(t)) \tag{A-19}$$

5b. For n ≥ 2 and for the conjugate gradient method, use

$$p^{(n)}(t) = \psi(0,t;q^{(n)}(t)) + \gamma^{(n)}\ p^{(n-1)}(t) \tag{A-20}$$

6. Calculate $\theta^{(n)}(x,t)$ using eqs. (A-10) to (A-12).

7. Calculate

$$\beta_D^{(n)} = \int_0^{t_f} [\theta^{(n)}(d,t)]^2 dt \tag{A-21}$$

8. Calculate

$$\beta_N^{(n)} = \int_0^{t_f} \psi(0,t;q^{(n)}(t))p^{(n)}(t)dt \tag{A-22}$$

9. Calculate

$$\beta^{(n)} = \beta_N^{(n)} \;/\; \beta_D^{(n)} \qquad \text{(A-23)}$$

10. Calculate

$$\gamma_D^{(n)} = \int_0^{t_f} [\psi(0,t;q^{(n)}(t))]^2 dt \qquad \text{(A-24)}$$

11. Calculate

$$q^{(n+1)}(t) = q^{(n)}(t) - \beta^{(n)} p^{(n)}(t) \qquad \text{(A-25)}$$

Go to step 1.

One relatively simple way to investigate the above algorithm is to use the convolution integral to calculate the T's, ψ's, and θ's. For the case of $T_o(x) = T_0$ and $q_L(t) = 0$, the temperature at any time t_M can be calculated using

$$T_M^{(n)} = T^{(n)}(d,t_M) = \int_0^{t_M} q^{(n)}(\lambda) \frac{\partial\phi(d,t_M-\lambda)}{\partial t} d\lambda + T_0 \qquad \text{(A-28a)}$$

where $\phi(d,t)$ is the temperature at $x = d$ caused by a heat flux q of 1 starting at $t = 0$. Eq. (A-28a) can be approximated as

$$T_M^{(n)} = \sum_{i=1}^{M} q_i^{(n)} \nabla\phi_{M-i+1} + T_0, \quad M=1,2,\ldots,M_f \tag{A-28b}$$

where

$$\nabla\phi_{M-i+1} = \phi_{M-i+1} - \phi_{M-i} \tag{A-28c}$$

$$M_f \, \Delta t = t_f \tag{A-28d}$$

and ϕ_{M-i} is ϕ evaluated at $x = d$ and time $t_{M-i} = (M-i)\Delta t$, with Δt being the time step. A simpler case than arbitrary d is for d being equal to L. For this case $\psi_k^{(n)}$ is approximated by

$$\psi_k^{(n)} = \sum_{i=1}^{M_f-k+1} \left[T_{k+i-1}^{(n)} - Y_{k+i-1}\right] \nabla\phi_i \quad , k = 1,2,\ldots,M_f \tag{A-29}$$

and

$$\theta_M^{(n)} = \sum_{i=1}^{M} p_i^{(n)} \nabla\phi_{M-i+1} \tag{A-30}$$

where

$$p_i^{(n)} = \psi_{M_f-i+1}^{(n)} + \gamma^{(n)} p_i^{(n-1)} \qquad \text{(A-31)}$$

An integral such as $\beta_D^{(n)}$ in eq. (A-21) can be approximated by

$$\beta_D^{(n)} = \sum_{i=1}^{M_f} [\theta_i^{(n)}]^2 \Delta t \qquad \text{(A-32)}$$

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